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Travelling waves in a nonlinearly suspended beam: some computational results and four open questions

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Travelling waves in a nonlinearly suspended beam: some computational results and four open questions

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A new nonlinear differential equation for a nonlinearly supported beam is proposed. Numerical evidence is presented on the existence, interaction, and spontaneous decomposition properties of travelling waves in such a beam. In particular, these waves seem to possess the ability to pass through each other like solitons, and more complicated shapes seem to decompose into simpler ones. At present, there is no mathematical explanation for these phenomena, and several open mathematical questions are posed.

1. Introduction

The purpose of this paper is to bring to the attention of the mathematical community a dramatic gap between a rich and complex collection of computational results on travelling waves in a simple model of a nonlinearly supported beam, and an almost complete absence of mathematical explanation for these results.

McKenna & Walter (1990) found travelling wave solutions to nonlinear beam equations on the real line of the form

$$u_{tt} + u_{xxxx} + u^+ - 1 = 0.$$

The search for travelling wave solutions of the form $u(x,t) = 1 + y(x - ct)$ then led to the ordinary differential equation

$$y'''' + c^2 y'' + (y+1)^+ - 1 = 0. \tag{1.1}$$

In McKenna & Walter (1990), this was shown to have solutions decaying to zero exponentially by 'brute force', namely, by solving the two *linear* ordinary differential equations explicitly for $y > -1$ and $y < -1$ and then matching them at $y = -1$. This purely calculus approach had serious limitations. The slightest perturbation in the nonlinearity rendered all these calculations invalid.

In Chen & McKenna (1997), this problem was solved in the sense that a qualitative proof based on the Mountain Pass lemma (Rabinowitz 1986; Brezis & Nirenberg 1991) was given which showed that for nonlinearities 'like' $(y+1)^+ - 1$, the ordinary differential equation

$$y'''' + c^2 y'' + f(y) = 0 \tag{1.2}$$

had at least one non-trivial solution. This was done by looking for critical points of the functional

$$I(y) = \int_R (\tfrac{1}{2}|y''|^2 - \tfrac{1}{2}c^2|y'|^2)\,\mathrm{d}x + \int_{y\leqslant -1} (|y| - \tfrac{1}{2})\,\mathrm{d}x + \int_{y>-1} (\tfrac{1}{2}|y|^2)\,\mathrm{d}x. \qquad (1.3)$$

It was shown that $y \equiv 0$ was a local minimum for I and that it was possible to choose other functions in $H^2(R^n)$ which had a lower value for I than $I(0)$. Modulo some technical details, this enabled us to use the Mountain Pass lemma with concentrated compactness to prove the existence of non-trivial solutions for a larger class of nonlinearities f.

Already, we come to the first problem.

Problem 1.1. *Can one prove that there are many solutions of the differential equation (1.3)?*

The calculations of McKenna & Walter (1990) suggest that there are many, possibly infinitely many, solutions, but all that can be proven is that there is at least one non-trivial solution.

Having found travelling wave solutions of (1.3), the next natural question was how these solutions behaved when used as initial data for the full initial value problem (1.1). In particular, it was natural to ask whether some or all of the travelling wave solutions were stable and persisted under slight perturbation in the initial data. If one was daring, one might hope for interesting interaction behaviour when two of these waves collided.

As described in Chen & McKenna (1997), we commenced such an investigation. However, almost immediately, we recognized a significant problem: the nonlinearity u^+ is not smooth and therefore any numerical procedure which treats this function may have large errors. This seemed to be the case in computations.

Accordingly, we decided to substitute for the nonlinearity $u^+ - 1$ a function which shared some of the qualitative behaviour but was analytic. The function we chose was $f(u) = \mathrm{e}^{u-1} - 1$ in the beam equation,

$$u_{tt} + u_{xxxx} + f(u) = 0,$$

giving rise to the nonlinear ordinary differential equation

$$y'''' + c^2 y'' + \mathrm{e}^y - 1 = 0. \qquad (1.4)$$

Again, $y \equiv 0$ is a solution and when one writes the corresponding variational problem, it corresponds to a local minimum of the functional. Again, it is easy to find functions at a lower elevation than the local minimum. However, carrying out the technical details of the previous existence proof has so far proven too difficult.

This brings us to our second open problem.

Problem 1.2. *Can one prove the existence of any non-trivial solutions of (2.1)?*

The next difficulty we faced with this nonlinearity was that of numerically finding solutions of (2.1). There were two solutions to this difficulty. The first one used was a numerical version for the Mountain Pass lemma, which has now been used for a variety of different nonlinear problems.

A second and much more efficient method was suggested to us by Champneys & Spence (1993), whose shooting method proved much faster in finding solutions of (2.1). Experiments described in this paper use solutions obtained by both methods, but primarily the second.

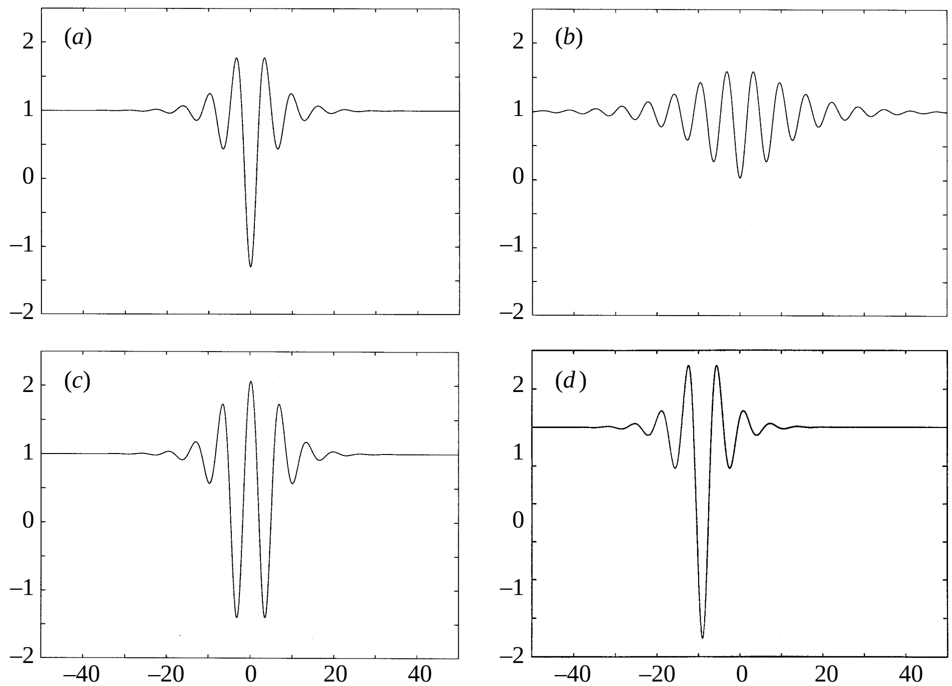

Figure 1. Travelling wave solutions of $u_{tt} + u_{xxxx} + \mathrm{e}^{u-1} - 1 = 0$ with speed.

The rest of this paper will consist of three sections. The first will briefly discuss the solutions of (2.1) found by the two methods, with a brief description of the Mountain Pass algorithm.

The next will discuss our observations of the stability properties of the waves when solved as an initial value problem for the beam equation. Finally, we describe some curious interaction properties of these waves, as yet entirely unexplained, including the facts that the waves appear to pass through each other, that their stability properties remain mysterious and that they occasionally can spontaneously decompose to simpler travelling waves of different speeds and magnitudes.

We should emphasize that these properties are of primarily mathematical interest, indicative of a deep and complex underlying structure. We do not expect to see such behaviour in modern-day suspension bridges.

2. Mountain pass algorithm

Based on the Mountain Pass theory, a numerical algorithm was developed in Choi & McKenna (1993).

It has been shown in Chen & McKenna (1997) that if a functional $I \in C^1(H^2, R)$ has a local minimum point e_1 and another point e_2 whose altitude $I(e_2)$ is lower than that of the minimum, then with some additional technicalities, we may conclude that the infimum of the maxima of I along all paths joining e_1 and e_2 is a critical value of I.

On a finite-dimensional approximating subspace, we take a piecewise linear path from e_1 to e_2 and find the maximum of I along that path. Then we deform the path

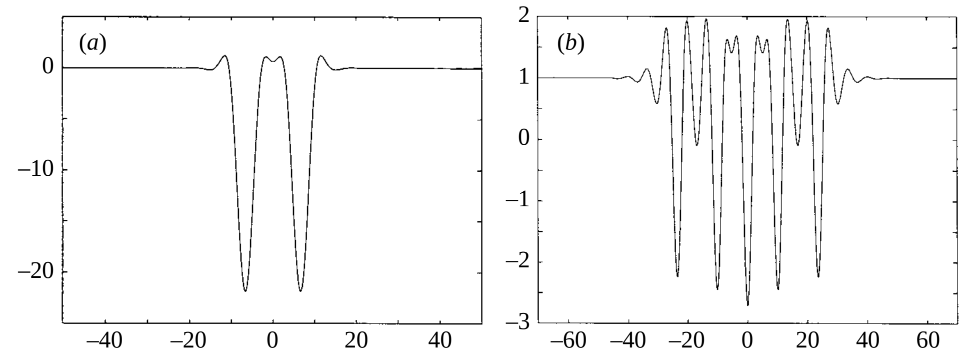

Figure 2. Travelling wave solutions of $u_{tt} + u_{xxxx} + e^{u-1} - 1 = 0$ found by continuation methods.

by pushing the maximum point in the direction of steepest descent. This step will be repeated until the critical point is reached.

Because of the fast growth of the function $e^{u-1} - 1$, we have encountered some difficulty in proving the existence of the travelling wave solutions for the equation

$$u_{tt} + u_{xxxx} + e^{u-1} - 1 = 0, \quad -\infty < x < \infty. \tag{2.1}$$

But by applying the Mountain Pass algorithm on a finite subinterval of R, we did obtain numerical solutions. The functional defined on $H^2(-L, L)$, whose critical points correspond to weak solutions of (2.1), is given by

$$I(y) = \frac{1}{2}\left[\int_{-L}^{L} (|y''|^2 - c^2|y'|^2)\,dx\right] + \int_{-L}^{L} F(y)\,dx, \tag{2.2}$$

where $F(y) = e^y - y - 1$.

The algorithm is similar to that for finding the Mountain Pass type critical points in the direct variational formulation of semilinear elliptic equations (Choi & McKenna 1993), the major difference being that the steepest descent direction is being sought in the H^2 norm in our case.

The conclusions of the search for solutions of (2.1) seem to indicate that the situation is rather similar to that of McKenna & Walter (1990). Solutions seem to exist in the range $0 < c < \sqrt{2}$. As the wave speed approaches $\sqrt{2}$, the solutions become highly oscillatory in nature, whereas when c approaches 0, they appear to go to infinity in amplitude.

There appear to be many different solutions for each wave speed and these can be obtained either by the Mountain Pass algorithm and varying the choice of the initial path by choosing different points for e_2, or by varying the shooting parameters.

We show some of the solutions obtained in figures 1 and 2. Notice that figures 1c and 1d are two different waves obtained from the Mountain Pass algorithm with the same wave speed.

3. Stability properties

Taking one of the numerical solutions we have obtained by applying the Mountain Pass algorithm as the initial function $u(x, 0)$, we solve the initial-boundary value problem

$$u_{tt} + u_{xxxx} + f(u) = 0$$

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by using a standard explicit central finite difference scheme (Strikwerda 1989) with the following non-reflecting boundary conditions:

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u(-L,t) = 0, \quad \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u(L,t) = 0\\ \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)^2 u(-L,t) = 0, \quad \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)^2 u(L,t) = 0.\end{aligned}\right\} \tag{3.1}$$

We then follow the solution of the initial value problem for some time in the hope of observing the stability properties of these waves. Figure 3 shows one of such travelling wave solutions.

We have also tested the stability of the solutions by disturbing the initial function $u(x,0)$ or the speed c of the wave, and the waves maintained the same basic shape over time. The results show that some of the solutions are quite stable in a sense that the nearby states remain nearby for future times (Strauss 1989).

The surprising conclusion seems to be that the stability of the simple wave is related to the shape. If a simple solitary wave is concave up about the centre, then it will be stable over a large time interval. On the other hand, if it is concave down, then solving the same initial value problem results in the wave rapidly losing its shape. Some of the tests are shown in figures 4 and 5.

In the case that a simple wave is unstable, an even more interesting phenomenon is observed in figure 5 (with periodic boundary conditions). The unstable wave seems to break into two simple stable waves, both concave up about their centres, one moves slower with larger amplitude, the other moves faster with smaller amplitude.

There are also multi-bump solutions—solutions which appear to be the linear combination of two waves that are separated apart. Two multi-bump solutions obtained by the continuation method are shown in figure 6.

So far it is not clear whether or not the stability properties of this kind of solutions are related to their shapes. Shown in figure 7 is a solution which appears to be a combination of two stable waves. This solution is stable for a long period of time, but is concave down at the central point.

Problem 3.1. *Prove that some solutions of (2.2) are stable. Account for why some of the solutions appear unstable. What is the connection between the shape of the solutions and their stability properties?*

To the best of the authors' knowledge, the only results available on stability which might be related are those of Levandosky (1998) and Sandstede (this volume), which are for a different type of nonlinearity.

4. Interaction properties

To observe the interaction of two travelling waves, we first obtain two solutions $y_1(t)$ and $y_2(t)$ by the Mountain Pass algorithm, then take the initial function $u(x,0)$ to be the linear combination of these two solutions:

$$u(x,0) = y_1(x+L) + y_2(x-L) + 1, \tag{4.1}$$

where L is quater the length of the experimental interval.

We can make the two waves travel toward each other by taking opposite signs for their wave speeds. It is shown in figure 8 that the two waves emerge apparently intact after the collision.

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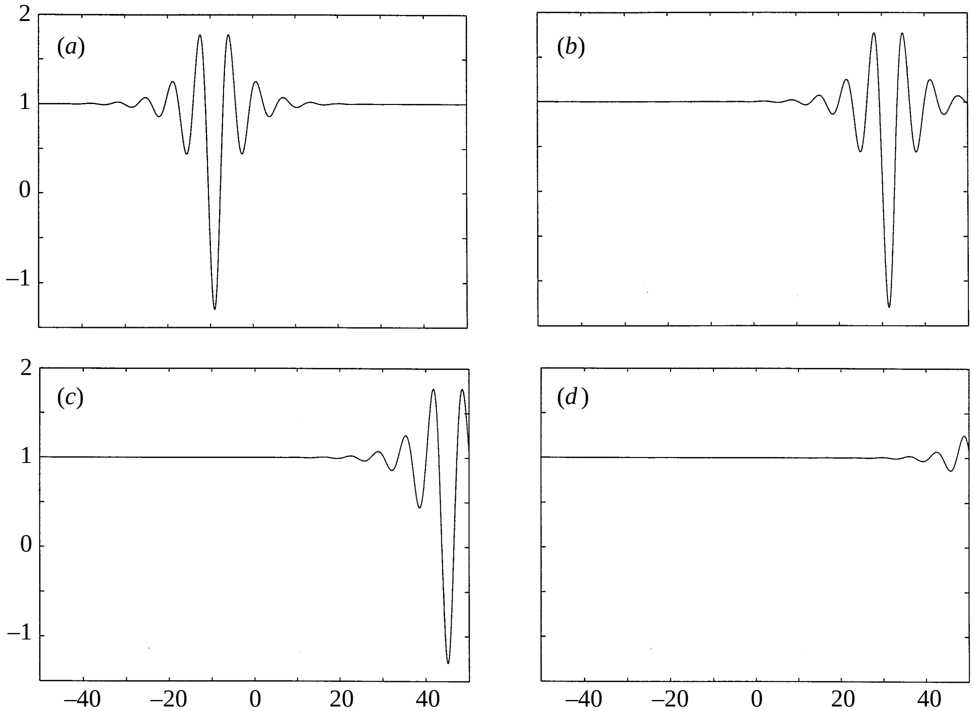

Figure 3. A travelling wave solution of $u_{tt} + u_{xxxx} + \mathrm{e}^{u-1} - 1 = 0$ with speed $c = 1.354$.

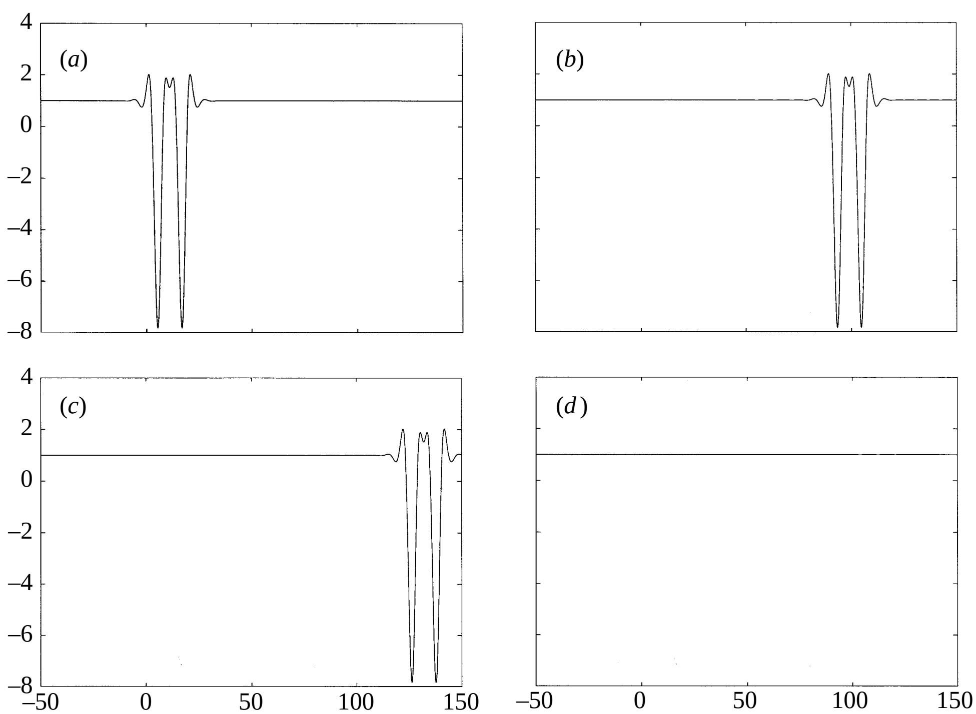

Figure 4. A stable solution with speed $c = 1.10$.

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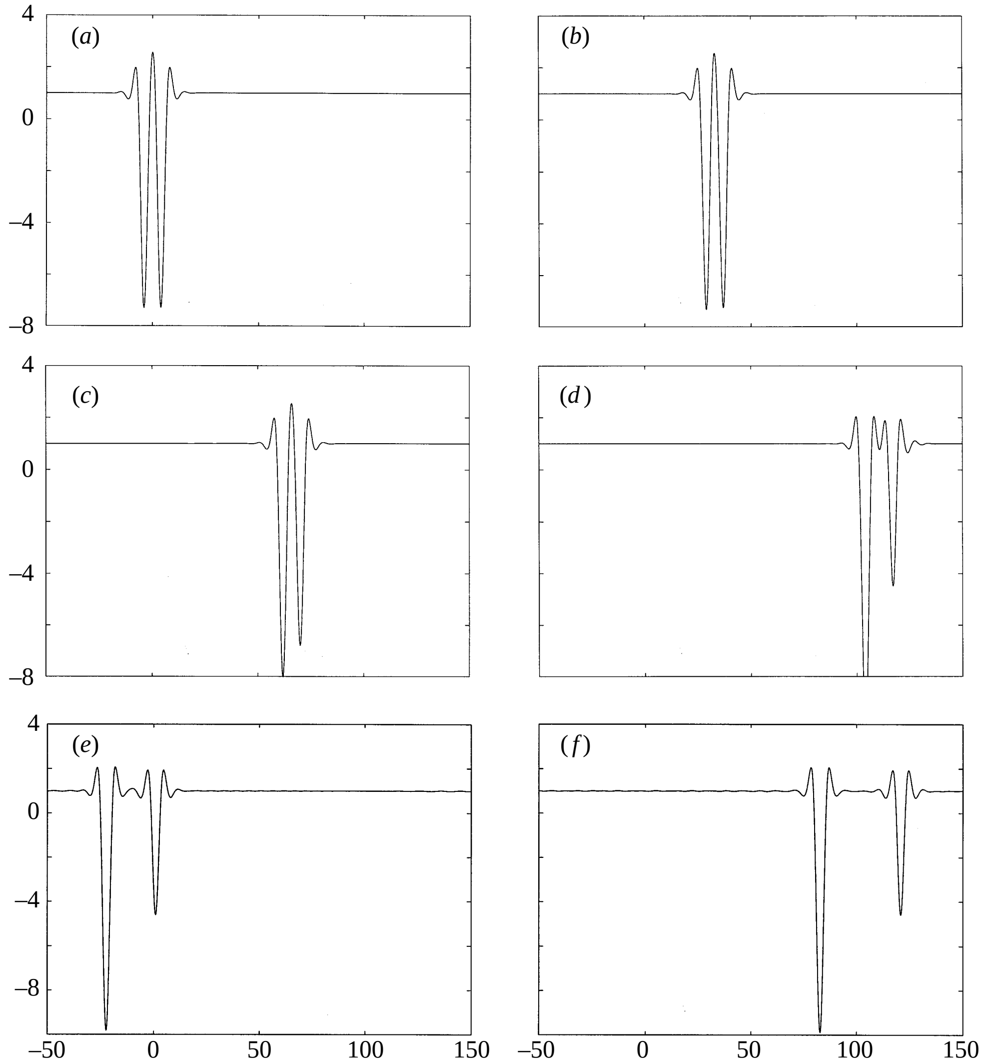

Figure 5. An unstable solution with speed $c = 1.10$.

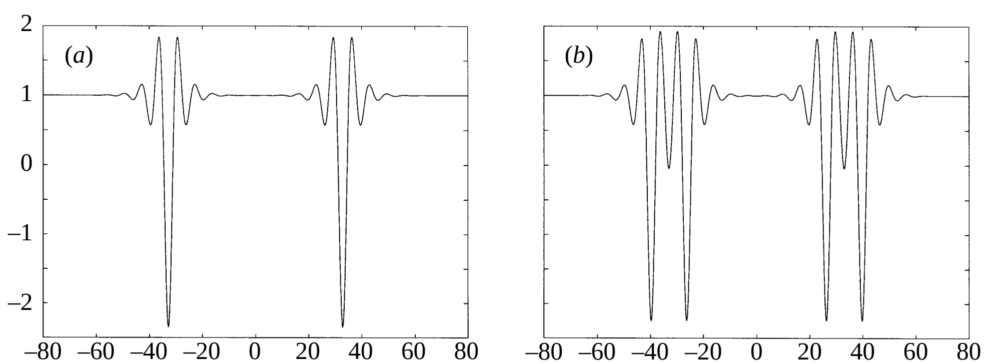

Figure 6. Solutions with speed $c = 1.30$.

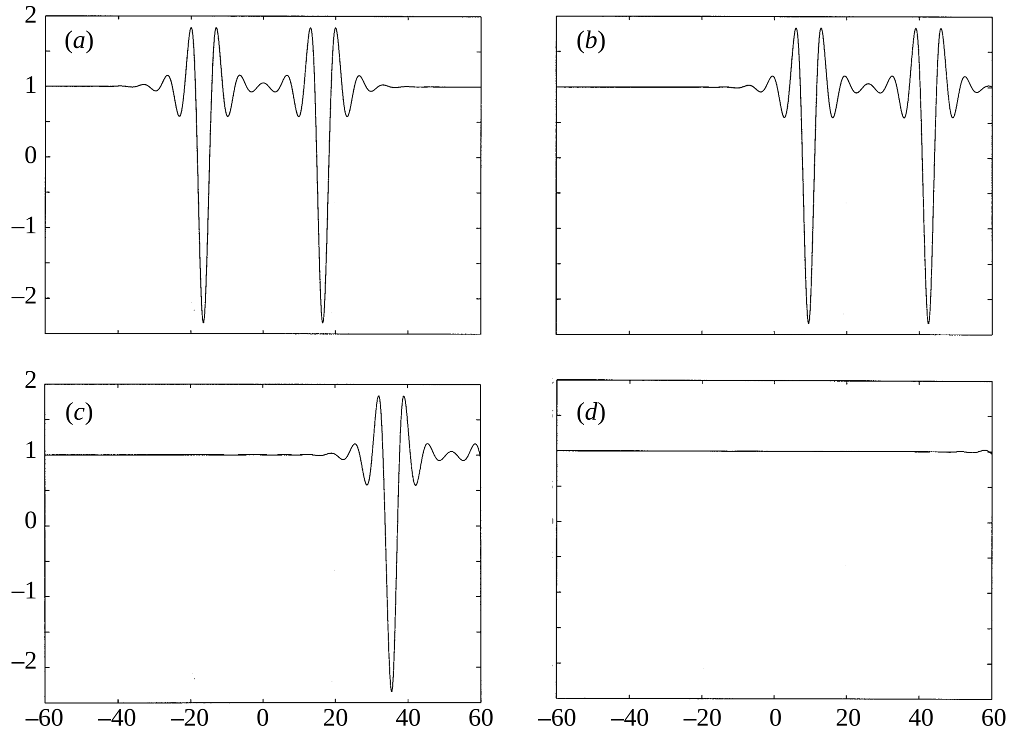

Figure 7. Solutions with speed $c = 1.30$.

We have not been able to observe this behaviour at wave speeds that are not close to $\sqrt{2}$. This may be because that as the wave speed gets smaller, the amplitude gets larger and that makes the initial value problem harder to solve accurately on a relatively coarse grid.

Usually, when one sees two nonlinear waves interacting in this manner, one expects to find additional conservation laws that are obeyed by solutions of the equation.

Problem 4.1. *Are there additional conserved quantities (apart from energy) for solutions of equation (2.2)? Can one give an explanation for the apparent property that solutions pass through each other?*

The authors are grateful to the referee who pointed out the connection with two other papers in the literature, namely Iooss & Pérouème (1993) and Grimshaw *et al.* (1994). An analysis of what happens in the neighbourhood for $c = \sqrt{2}$ was carried out using normal form analysis for some related problems including the Korteweg–de Vries equation.

5. Summary

We summarize the results of this paper. Probably the most striking result is the observation that these travelling waves pass through each other without losing their stability. This type of interaction of nonlinear waves has only previously been seen in completely integrable systems.

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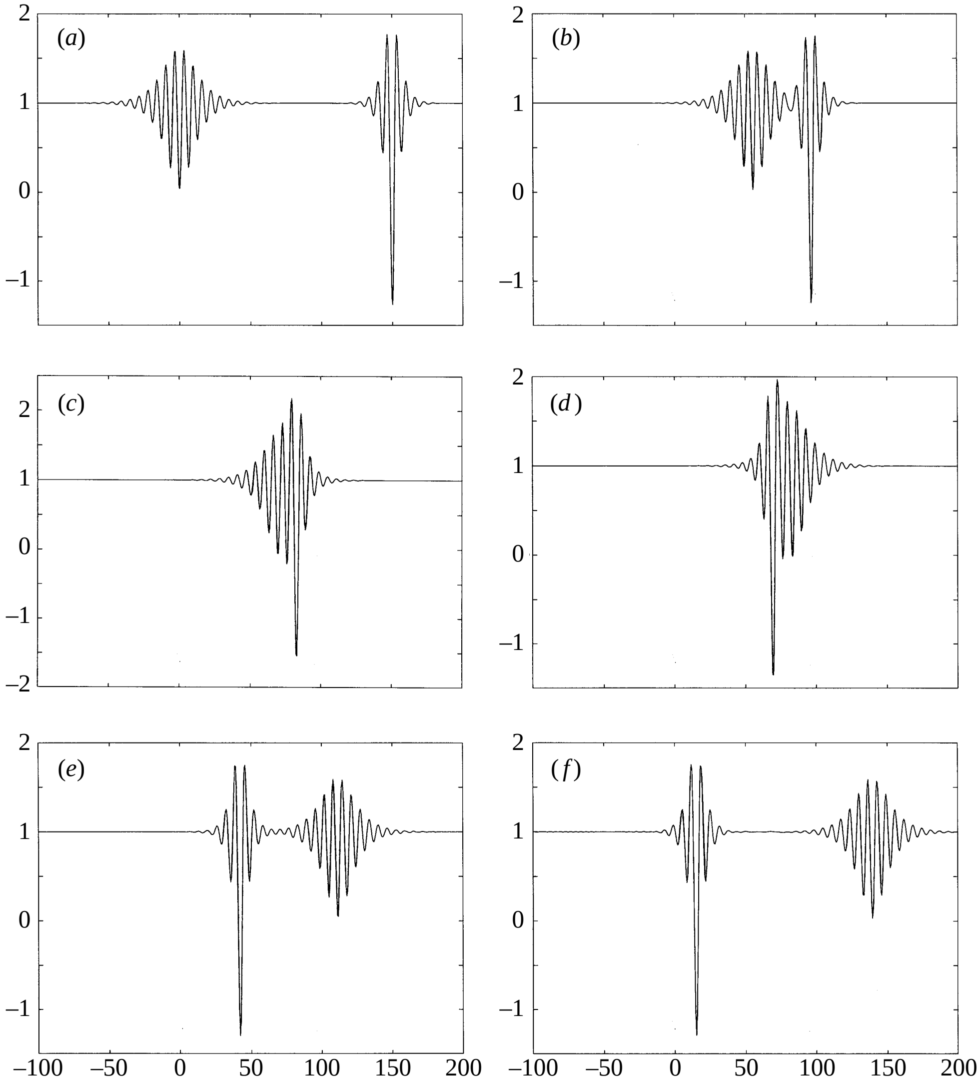

Figure 8. Interaction of two traveling waves of $u_{tt} + u_{xxxx} + \mathrm{e}^{u-1} - 1 = 0$.

This leads us to wonder whether we are in the presence of such a system†. Among experts, there seems to be little agreement about how to go about determining whether this is the case, other than be searching for additional conservation laws, in which we have so far been unsuccessful. Is there a methodical way to test for additional conservation laws?

Alternatively, this remarkable property of passing through each other may be related to some entirely different deep underlying structure, whose nature is still not apparent, but not of the completely integrable variety.

The experiment shown in figure 5 is also puzzling. Apparently, the more complex many-noded wave can exist for a while, and then spontaneously decompose into two

† One internationally known figure conjectured that equation (2.1) is connected to the Toda lattice, but the nature of the connection is as yet not clear.

simpler waves with less structure. While we are not proposing that a nonlinear beam equation should model elementary particle interactions, this result nonetheless also hints at some deeper fundamental property of these waves.

Just as earlier results on completely integrable systems were preceded by a period where the numerical results were unexplained, we believe this may be the case here, and that in the near future we shall have significant mathematical progress on the existence, stability, interactions and decompositions of these remarkable waves.

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